



Letter to the Editor

Superaccurate finite element eigenvalue computation

I. Fried*, M. Chavez

Department of Mathematics, Boston University, 111 Cummington Street, Boston, MA 02215, USA

Received 30 September 2003; accepted 9 October 2003

1. Introduction

The consistent finite element method for the computation of critical values is theoretically sound and also provides an assured upper bound on the lowest eigenvalue. But the price of consistency is the general algebraic eigenvalue $Kx = \lambda Mx$ that is computationally more complicated and costly than the special $Kx = \lambda x$ since it may necessitate the transformations $M^{-1}Kx = \lambda x$, or $L^{-1}KL^{-T}y = \lambda y$, $M = LL^T$, $y = L^T x$, that destroy the sparsity of the global stiffness matrix K , exacting thereby a heavy toll on the linear algebraic computation. An upper bound on the fundamental eigenvalue λ is theoretically interesting but may be of limited practical value particularly since a lower bound on λ is so hard to come by. Mass lumping producing a diagonal mass matrix M is, therefore, an attractive option for the engineer confronted with large complex systems. It can be looked upon as replacing the continuous mass distribution by lumps concentrated at the nodes, and if done correctly [1] preserves the accuracy of the finite element approximation. Such lumping may lead to an underestimate of the fundamental eigenvalue and a convergence from below as the mesh is appropriately refined.

If it is so that the consistent finite element formulation leads to an overestimation of λ and the lumped finite element formulation leads to an underestimation of λ , then it stands to reason that an intermediate formulation should exist that is accurately superior to both formulations. It is the purpose of this note to show, theoretically at first and numerically later on, that a superconvergent finite element formulation indeed exists for the eigenproblem discretized with low as well as high order finite elements in one or more dimensions.

2. Two-nodes string element

We start with looking at the simplest most transparent problem, that of the vibrating unit string, described by the boundary value problem

$$u'' + \lambda u = 0, \quad 0 < x < 1, \quad u(0) = u(1) = 0 \quad (1)$$

*Corresponding author. Tel.: +1-617-353-2560; fax: +1-617-353-8100.

E-mail address: if@math.bu.edu (I. Fried).

in which $\lambda = \pi^2$ is the lowest eigenvalue, and $u = \sin \pi x$ the corresponding eigenfunction. It is also customary to write $\lambda = \omega^2$ with ω being the fundamental frequency of the system. Eigenproblem (1) can be expressed variationally in the form

$$\lambda = \min_{\tilde{u}} \frac{\int_0^1 \tilde{u}'^2 dx}{\int_0^1 \tilde{u}^2 dx}, \quad \tilde{u}(0) = \tilde{u}(1) = 0 \tag{2}$$

in which trial function $\tilde{u} = 0$ is required to be at least continuous in $0 \leq x \leq 1$. The essence of finite elements consists of constructing \tilde{u} as a piecewise polynomial.

Let the typical finite element extend over $x_1 \leq x \leq x_1 + h$. The quadratic form definitions

$$\int_{x_1}^{x_1+h} \tilde{u}'^2 dx = u_e^T k_e u_e, \quad \int_{x_1}^{x_1+h} \tilde{u}^2 dx = u_e^T m_e u_e \tag{3}$$

for the element nodal vector u_e produce for a piecewise linear \tilde{u} the element matrices

$$k_e = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad m_e = \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad m_e = \frac{h}{2} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \tag{4}$$

in which k_e is the element stiffness matrix, and in which the first element mass matrix m_e is the consistent matrix for which $u_e^T m_e u_e$ is true for $\tilde{u} = 1$ and $\tilde{u} = x$, while the second element mass matrix m_e in Eq. (4) is the lumped matrix with $u_e^T m_e u_e$ true only for $\tilde{u} = 1$.

Let the interval $0 \leq x \leq 1$ be divided into $n + 1$ sections of size $h = 1/(n + 1)$ with nodes labeled $0, 1, 2, \dots, n, n + 1$. Assembly of the linear finite elements over this mesh using the lumped mass matrix leads to the typical finite difference equation

$$u_j - 2u_{j+1} + u_{j+2} + \omega^2 h^2 u_{j+1} = 0, \quad u_0 = u_{n+1} = 0. \tag{5}$$

This recursive equation is solved by $u_j = z^j$ provided that z satisfies the characteristic equation

$$z^2 + z(-2 + \omega^2 h^2) + 1 = 0 \tag{6}$$

having complex conjugate roots under the condition that

$$\omega^2 h^2 < 4. \tag{7}$$

Then

$$z = 1 - \frac{1}{2} \omega^2 h^2 \pm i h \omega \sqrt{1 - \frac{1}{4} \omega^2 h^2} \tag{8}$$

in which $i^2 = -1$. Since $|z| = 1$ we may write the roots of Eq. (8) as $z = \cos \theta \pm i \sin \theta$ and have that $u_j = c_1 z_1^j + c_2 z_2^j$ with c_1 and c_2 depending on the boundary conditions. Now

$$u_j = (c_1 + c_2) \cos j\theta + i(c_1 - c_2) \sin j\theta = A \cos j\theta + B \sin j\theta \tag{9}$$

and we have that $A = 0$ since $u_0 = 0$. The other boundary condition $u_{n+1} = 0$ leads to $B \sin(n + 1)\theta = 0$, and for a non-trivial solution we discount $B = 0$ and require instead that

$$(n + 1)\theta = \pi \quad \text{or} \quad \theta = \pi h. \tag{10}$$

From Eq. (8) we have that

$$\cos \pi h = 1 - \frac{1}{2} \omega^2 h^2 \tag{11}$$

or

$$\omega^2 = \frac{2}{h^2}(1 - \cos \pi h).$$

Power series expansion of $\cos \pi h$ results in

$$\lambda = \pi^2 \left(1 - \frac{1}{12} \pi^2 h^2 \pm \dots\right), \tag{12}$$

clearly indicating that λ in Eq. (12) is an underestimation of π^2 of accuracy $O(h^2)$.

Assembly of the linear finite elements with the consistent mass matrix results in the typical finite difference equation

$$u_j + 2u_{j+1} + u_{j+2} + \frac{1}{6} \omega^2 h^2 (u_j + 4u_{j+1} + u_{j+2}) = 0 \tag{13}$$

with the associated characteristic equation

$$z^2 + 2 \frac{-6 + 2\omega^2 h^2}{6 + \omega^2 h^2} z + 1 = 0. \tag{14}$$

When complex, the roots of Eq. (14) may again be written as $z = \cos \theta \pm i \sin \theta$, and here

$$\cos \pi h = \frac{6 - 2\omega^2 h^2}{6 + \omega^2 h^2} \tag{15}$$

so that

$$\omega^2 = \frac{6}{h^2} \frac{1 - \cos \pi h}{2 + \cos \pi h}. \tag{16}$$

Power series expansion reduces Eq. (16) to

$$\omega^2 = \pi^2 \left(1 + \frac{1}{12} \pi^2 h^2 + \frac{1}{360} \pi^4 h^4 + \dots\right) \tag{17}$$

and now, as expected, the computed ω^2 is an overestimation of π^2 of the same accuracy $O(h^2)$.

3. Optimal element mass distribution

Instead of Eq. (13) we propose to write the general finite difference approximation

$$u_j - 2u_{j+1} + u_{j+2} + \omega^2 h^2 (\alpha_0 u_j + \alpha_1 u_{j+1} + \alpha_0 u_{j+2}) = 0 \tag{18}$$

with the intent of finding the best masses α_0 and α_1 for highest eigenvalue accuracy.

The characteristic equation of finite differences scheme (18) is

$$z^2 + 2z \frac{-1 + \frac{1}{2} \alpha_1 \omega^2 h^2}{1 + \alpha_0 \omega^2 h^2} + 1 = 0 \tag{19}$$

and by the previous arguments we have here that

$$\cos \pi h = \frac{2 - \alpha_1 \omega^2 h^2}{2 + 2\alpha_0 \omega^2 h^2} \tag{20}$$

or

$$\omega^2 = \frac{1}{\alpha_0 h^2} \frac{1 - \cos \pi h}{\beta + \cos \pi h}, \quad \beta = \frac{\alpha_1}{2\alpha_0}. \tag{21}$$

We write $\beta = \alpha_1/(2\alpha_0)$, $x = \pi h$, and have by power series expansion that

$$\frac{1 - \cos x}{\beta + \cos x} = x^2 \left[\frac{1}{2(1 + \beta)} + \frac{5 - \beta}{24(1 + \beta)^2} x^2 + O(x^4) \right] \tag{22}$$

so that

$$\omega^2 = \pi^2 \left[\frac{1}{2\alpha_0 + \alpha_1} + \frac{10\alpha_0 - \alpha_1}{12(2\alpha_0 + \alpha_1)^2} x^2 + O(x^4) \right] \tag{23}$$

which with the choice $2\alpha_0 + \alpha_1 = 1$, $10\alpha_0 - \alpha_1 = 0$ or, $\alpha_0 = \frac{1}{12}$, $\alpha_1 = \frac{10}{12}$, results in

$$\omega^2 = \pi^2 \left(1 - \frac{\pi^4 h^4}{240} \pm \dots \right) \tag{24}$$

with a superior asymptotic accuracy $O(h^4)$.

4. A finite element interpretation

The optimal finite difference scheme

$$u_j + 2u_{j+1} + u_{j+2} + \frac{\omega^2 h^2}{12}(u_j + 10u_{j+1} + u_{j+2}) = 0 \tag{25}$$

can be interpreted as an equation produced by a finite element discretization with the element mass matrix

$$m_e = \frac{h}{12} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \tag{26}$$

which is but a special case of

$$m_e(\gamma) = \frac{h}{6} \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \tag{27}$$

with $\gamma = \frac{1}{2}$. Otherwise the element mass matrix in Eq. (27) is such that $m_e(\gamma = 0)$ is the consistent element mass matrix and $m_e(\gamma = 1)$ is the lumped.

5. Three-nodes string element

A similar theoretical analysis for the quadratic string element is cumbersome and we revert to a numerical argument.

The element matrices for a three-nodes quadratic element of size $2h$ are

$$k_e = \frac{1}{6h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}, \quad m_e = \frac{h}{15} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad m_e = \frac{h}{3} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 1 \end{bmatrix} \tag{28}$$

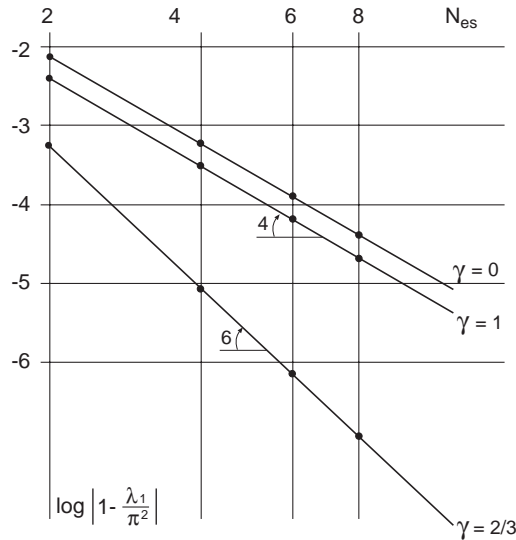


Fig. 1. Convergence of fundamental eigenvalue λ_1 of a unit string and the element mass matrix in Eq. (29).

with the lumped m_e producing the correct kinetic energy for $\tilde{u} = 1$ and $\tilde{u} = x$, but not for $\tilde{u}^2 = x^2$, as the consistent m_e does. We write the general m_e , true for a linear \tilde{u} , as

$$m_e(\gamma) = \frac{h}{15} \left(\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \gamma \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \right) \tag{29}$$

so that $m_e(\gamma = 0)$ is the consistent element mass matrix and $m_e(\gamma = 1)$ is the lumped element mass matrix. Does a value of γ exist for which convergence of the fundamental eigenvalue λ is superfast? Computations confirm, as shows in Fig. 1, that $\gamma = \frac{2}{3}$ leads to eigenvalue convergence at rate 6 as opposed to convergence at rate 4 for both $\gamma = 0$ and 1.

6. Three-nodes triangular membrane element

We shall now look at the discretization of boundary value problem

$$u_{xx} + u_{yy} + \lambda u = 0 \text{ in domain } D \text{ with } u = 0 \text{ on boundary } S \tag{30}$$

which is variationally equivalent to

$$\lambda = \min_{\tilde{u}} \frac{\int \int (\tilde{u}_x^2 + \tilde{u}_y^2) dx dy}{\int \int \tilde{u}^2 dx dy}, \tag{31}$$

where the double integration extends over D , for \tilde{u} being at least continuous over the membrane and with $\tilde{u} = 0$ on the boundary where $u = 0$ prevails. We define the element stiffness and mass matrices of a triangular membrane element Δ as holding the coefficients

in the quadratic forms

$$\int_A \int (\tilde{u}_x^2 + \tilde{u}_y^2) dx dy = u_e^T k_e u_e, \quad \int_A \int \tilde{u}^2 dx dy = u_e^T m_e u_e. \tag{32}$$

For a linear \tilde{u} we obtain from Eq. (32) the element stiffness matrix

$$k_e = \frac{1}{8A} \left(L_1^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} + L_2^2 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} + L_3^2 \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} \right) \tag{33}$$

in which L_1, L_2, L_3 are the sides of the triangle and A its area. As for the mass matrices of the triangular three-nodes membrane element we obtain, from Eq. (32),

$$m_e = \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad m_e = \frac{A}{12} \begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix} \tag{34}$$

that we combine into

$$m_e(\gamma) = \frac{A}{12} \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \right) \tag{35}$$

so that $m_e(\gamma = 0)$ is the consistent element mass and $m_e(\gamma = 1)$ is the lumped.

We use this element to discretize an equilateral triangular membrane of unit sides that is known [2] to have a fundamental eigenvalue of $\lambda = 16\pi^2/3$. Fig. 2 shows the convergence of λ as the number of elements per side N_{es} is increased. Computation is shown for $\gamma = 0, \gamma = 1$, and $\gamma = \frac{1}{2}$, and indeed convergence is of $O(N_{es}^{-2})$ for both $\gamma = 0$ and $\gamma = 1$, but jumps to $O(N_{es}^{-4})$ with $\gamma = \frac{1}{2}$.

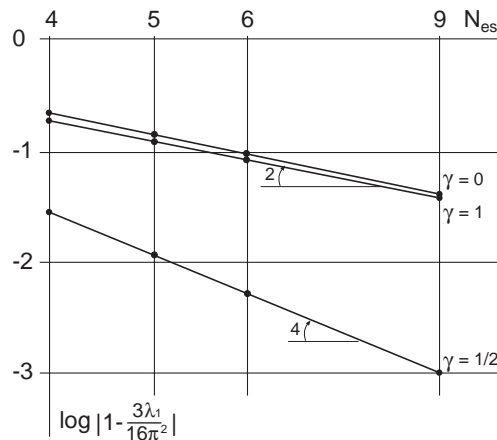


Fig. 2. Convergence of fundamental eigenvalue λ_1 of a unit triangular membrane and the element mass matrix in Eq. (35).

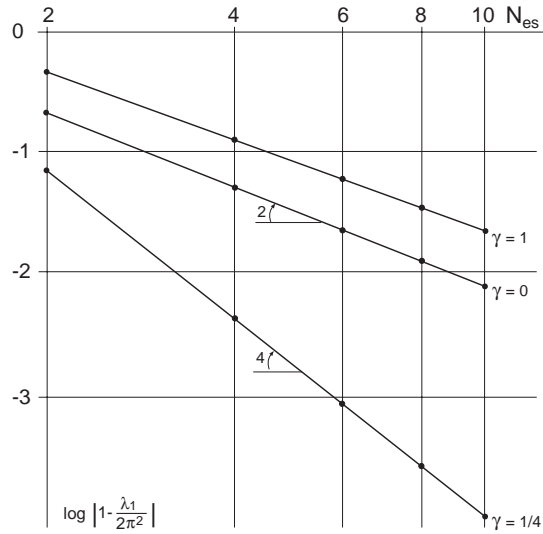


Fig. 3. Convergence of fundamental eigenvalue λ_1 of a unit square membrane and the element mass matrix in Eq. (36).

7. Four-nodes square membrane element

The next element we consider is a four-nodes square of side h , with node 4 being opposite node 1, and node 3 being opposite node 2. Its element stiffness and mass matrices are

$$k_e = \frac{1}{6} \begin{bmatrix} 4 & -1 & -1 & -2 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -2 & -1 & -1 & 4 \end{bmatrix},$$

$$m_e(\gamma) = \frac{h^2}{36} \left(\begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} + \gamma \begin{bmatrix} 5 & -2 & -2 & -1 \\ -2 & 5 & -1 & -2 \\ -2 & -1 & 5 & -2 \\ -1 & -2 & -2 & 5 \end{bmatrix} \right) \quad (36)$$

with $m_e(\gamma = 0)$ being the consistent element mass matrix and $m_e(\gamma = 1)$ being the element lumped mass matrix. We use this element to discretize a unit square membrane for which $\lambda_1 = 2\pi^2$.

Fig. 3 shows the error in the computed first eigenvalue of the square membrane versus the number of elements per side N_{es} on a logarithmic scale. Indeed, for $\gamma = 0$ and 1, the error in λ is $O(N_{es}^{-2})$, whereas for $\gamma = \frac{1}{4}$, the accuracy dramatically improves to $O(N_{es}^{-4})$.

8. Conclusions

It has been shown that the finite element mass matrix may be written as a linear function of parameter γ , $m_e = m_1 + \gamma m_2$ so that $m_e(\gamma = 0)$ is the consistent element mass matrix and $m_e(\gamma = 1)$

is the lumped element mass matrix. Convergence to the fundamental eigenvalue λ is from above if $\gamma = 0$ and is from below if $\gamma = 1$. An intermediate value of λ produces an element mass matrix with which superaccurate eigenvalues are computed. This is demonstrated for the two- and three-nodes string element, for the three-nodes triangular membrane element, and for the four-nodes rectangular membrane element.

References

- [1] I. Fried, D.S. Malkus, Finite element mass matrix lumping by numerical integration with no convergence loss, *International Journal of Solids and Structures* 11 (1975) 461–466.
- [2] J.R. Kutter, V.G. Sigillito, Eigenvalues of the Laplacian in two dimensions, *SIAM Review* 26 (1984) 163–193.